$$3\mu^2a^2=[\mu a+Q\sin(\phi-\theta)]^2+[\mu b+Q\sin(\theta+\tfrac{1}{2}\beta-\phi)]^2$$
$$+2[\mu a+Q\sin(\phi-\theta)][\mu b+Q\sin(\theta+\tfrac{1}{2}\beta-\phi)]\cos\tfrac{1}{2}\beta\ldots(4).$$

Eliminating ϕ between (3) and (4) gives an equation to determine θ.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

152. Proposed by H. S. VANDIVER, Bala, Pa.

When p is a prime of the form $5n\pm1$ then there is a positive integer a such that $a^2\equiv5(\text{mod } p)$. Show that $\left(\frac{a\pm1}{p}\right)=\pm\left(\frac{-2a}{p}\right)$, according as p is of the form $5n+1$ or $5n-1$.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

A general expression involving all cases is, apparently, not easily deduced. The following cases hold.

(1) Let $n=2$. Then $p=5n+1=11$. $a=4$, $a=p-4=7$, $a=2p-7=15$, $a=3p-15=18$, etc.

(2) Let $n=4$. Then $p=5n-1=19$, $a=9$, $a=p-9=10$, $a=2p-10=28$, $a=3p-28=29$, etc.

(3) Let $n=6$. Then $p=5n+1=31$, $a=6$, $a=p-6=25$, $a=2p-25=37$, $a=3p-37=56$, etc.

(4) Let $n=8$. Then $p=5n+1=41$, $a=13$, $a=p-13=28$, $a=2p-28=54$, $a=3p-54=69$, etc.

(5) Let $n=12$. Then (b) $p=5n+1=61$; (c) $p=5n-1=59$.

(b) $a=26$, $a=p-26=35$, $a=2p-35=87$, $a=3p-87=96$, etc.

(c) $a=8$, $a=p-8=51$, $a=2p-51=67$, $a=3p-67=110$, etc.

For every value of n that makes $5n\pm1$ a prime, we can find values for a satisfying the condition. It is also easy to see that a can have an infinitude of values for each case.

In (1), (3), (4), (5)(b), $(a+1)^{\frac{1}{2}(p-1)}=(-2a)^{\frac{1}{2}(p-1)}\equiv-1(\text{mod. } p)$.

In (2), (5)(c), $(a-1)^{\frac{1}{2}(p-1)}=-(-2a)^{\frac{1}{2}(p-1)}\equiv1(\text{mod. } p)$.

$\left(\frac{a\pm1}{p}\right)=\pm\left(\frac{-2a}{p}\right)$, in the cases examined, according as $p=5n\pm1$.

A general solution is desired. ED. F.

152. Proposed by H. S. VANDIVER, Bala, Pa.

Prove geometrically:

$\sum_{n=1}^{\frac{1}{2}(p-1)}\left[\frac{n^2}{p}\right]=\frac{p-3}{4}\cdot\frac{p-1}{2}-\sum_{n=1}^{\frac{1}{4}(p-4}\left[\sqrt{np}\right]$, where $p\equiv3(\text{mod. } 4)$ and $\left[\frac{k}{p}\right]$ represents the greatest integer in k/p.

Remarks by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

In the second term of the second member of the equation, $\frac{1}{4}(p-4)$ should be $\frac{1}{4}(p-3)$. The equation is generally but not universally true as is shown by induction in what follows.

Let $p=4m+3$, then $\frac{1}{2}(p-1)=2m+1$, $\frac{1}{4}(p-3)=m$,

$$\sum_{n=1}^{\frac{1}{2}(p-1)}\left[\frac{n^2}{p}\right]=\frac{p-3}{4}\cdot\frac{p-1}{2}-\sum_{n=1}^{\frac{1}{4}(p-3)}[\sqrt{(np)},$$

or as follows:

$$\begin{array}{rr} & A=\ B-C, \\ m=2, & 3=\ 10-7, \\ m=3, & 7=\ 21-14, \\ m=4, & 11=\ 36-25, \\ m=5, & 18=\ 55-37, \\ m=7, & 34=105-71. \end{array}$$

If one of the $[n^2/p]$ is an exact quotient, and hence one of the $[\sqrt{(np)}]$ rational, the equation is $A=1+B-C$.

$$m=6,\ p=27,\ [9^2/p]=3,\ \sqrt{(3\times27)}=9,$$
$$m=15,\ p=63,\ 21^2/p=7,\ \sqrt{(7\times63)}=21.$$

$\therefore A=1+B-C$, $m=6\ldots25=1+78-54$, $m=15\ldots153=1+465-313$.

If two of the $[n^2/p]$ are exact quotients, and hence two of the $[\sqrt{(np)}]$ rational, the equation becomes $A=2+B-C$.

$$m=18,\ p=75,\ 15^2/p=3,\ \sqrt{(3\times p)}=15,\ 30^2/p=12,\ \sqrt{(12\times p)}=30.$$

$\therefore A=2+B-C$ becomes $219=2+666-449$ for $m=18$. $A=t+B-C$ is the true universal equation.

The geometric proof in this solution is wanting. Who can produce it? ED. F.

155. Proposed by PROF. R. D. CARMICHAEL, Anniston, Alabama.

If p and q are primes and m and n are any integers, find the cases in which the equation $p^m-q^n=1$ may be satisfied.

Remarks by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Some values, found by inspection, are given in the following table: